Scattering approach to fidelity decay in closed systems and parametric level correlations

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Abstract. This paper is based on recent work which provided an exact analytical description of scattering fidelity experiments with a microwave cavity under the variation of an antenna coupling [Köber et al., Phys. Rev. E 82, 036207 (2010)]. It is shown that this description can also be used to predict the decay of the fidelity amplitude for arbitrary Hermitian perturbations of a closed system. Two applications are presented: First, the known result for global perturbations is re-derived, and second, the exact analytical expression for the perturbation due to a moving S-wave scatterer is worked out. The latter is compared to measured data from microwave experiments, which have been reported some time ago. Finally, we generalize an important relation between fidelity decay and parametric level correlations to arbitrary perturbations.

1. Introduction

During the last decade, approximately, considerable efforts have been dedicated to the quantitative prediction of the fidelity decay in chaotic/diffusive quantum systems and classical wave systems [1, 2, 3, 4] (see also [5] and references therein). A very successful approach has been based on random matrix theory, adopting the so called Bohigas-Giannoni-Schmit conjecture [6]. Applied to the current setting, it suggests that quantum systems with chaotic classical counterpart ("chaotic quantum systems" for short) as well as diffusive wave systems show a universal response to perturbations which can be calculated within an appropriate random matrix model [7]. The first exact analytical results in this respect have been obtained by Stöckmann and Schäfer [8, 9] using super-symmetry techniques similar to those for the calculation of correlation functions between scattering matrix elements in [10]. More recently, exact analytical results have also been found for scattering systems, where the fidelity amplitude, an expectation value, is replaced by the "scattering fidelity", which is a product of two transition amplitudes [11, 12]. These results, published in [13], have been obtained by a simple but powerful modification of the Verbaarschot-Weidenmüller-Zirnbauer ("VWZ" for short) formula from [10]. We call this approach the "scattering approach to fidelity".

As shown first by Kohler et al. [14], for a global perturbation of a completely diffusive system, the fidelity amplitude can also be calculated from the parametric level correlations. Subsequent generalizations have been discussed in [15, 16] and [17]. Originally, parametric level correlations have been introduced in the area of disordered systems with diffusive dynamics [18, 19]. At that moment, they have been considered a universal signature of chaotic/diffusive dynamics, where the functional does not depend on the perturbation applied. However, in [20] it was shown that certain types of perturbations may lead to pronounced deviations. Thereby, it became clear that the "universal" theoretical prediction of Simons and Altshuler [19] only applies to global perturbations, not local ones, where the perturbation operator has only a few eigenstates with non-zero eigenvalues. The perturbation due to the displacement of a small scatterer discussed in [20] is precisely of that latter type.

The purpose of the present paper is twofold. Firstly, we use the scattering fidelity approach from [13] to derive exact analytical expressions for the fidelity decay of chaotic/diffusive wave systems in the presence of completely general Hermitian perturbations. This allows us, to re-derive the known result for the decay of the fidelity amplitude due to a global perturbation [8, 9]. In the second example, we use it to derive an exact analytical expression for the decay of the fidelity amplitude due to the displacement of a S-wave scatterer (local perturbation). In [21], the decay of the fidelity amplitude has been obtained from experimental for such a case [20].

Secondly, we generalize the relation between the fidelity amplitude and the parametric level correlations from [14], to arbitrary perturbations. To do so, we compare the analytical expression for the parametric level correlations [22] and its analogue for the fidelity amplitude for general perturbations. Our result is important, as it allows

to calculate the fidelity amplitude for spectral data, only. It thereby shows that the fidelity amplitude is a basis independent – which is surprising taking into account that the perturbation may be completely arbitrary. From a practical point of view, one may easily find situations, where the measurement of level spectra and their variation under certain perturbations is easier and more accurate than any fidelity measurement. The new relation shows, that a measurement of parametric level correlations provides exactly the same information about an a priori unknown perturbation than a fidelity measurement.

The present paper is organized as follows: In the next section, we follow [13] to describe the connection between scattering fidelity [11] and scattering matrix correlation functions as considered in [10]. We use this connection to derive an exact analytical expression for the fidelity amplitude valid for arbitrary perturbations. In section 3 we discuss the differences between local and global perturbations, and we use our general formula to re-derive the known result for a global perturbation. In the remaining part of that section, we calculate the fidelity amplitude in the case of a moving scatterer and compare the resulting theoretical prediction to experimental data from [21]. In section 4, we evaluate the general integral expression for the fidelity amplitude in the perturbative/long time limit. In section 5 we generalize the relation between parametric level correlations and the fidelity amplitude to arbitrary perturbations. Conclusions are presented in section 6.

2. Scattering approach to fidelity

In this section, we introduce the central quantity of this work, the fidelity amplitude of a closed quantum or classical wave system, with quantum chaotic or diffusive dynamics. We assume that random matrix theory can be used to describe the fidelity decay. While the first part contains some general statements about fidelity and the random matrix models used, the second part describes the description of the algebraic scattering model to which the fidelity problem is mapped. This mapping, introduced in [13] provides an exact analytical description of the fidelity decay.

2.1. Fidelity

The fidelity and the fidelity amplitude for a Hamiltonian $H_\alpha = H_0 + W_\alpha$ with perturbation W_α are defined as

$$F(t) = |f(t)|^2 \,, \qquad f(t) = \langle a|\, \mathrm{e}^{2\pi \mathrm{i} H_\beta\, t}\, \mathrm{e}^{-2\pi \mathrm{i} H_\alpha\, t}\, |a\rangle \tag{1}$$

where $|a\rangle$ is the initial state and W_α is the perturbation depending on an external parameter α. We assume that the energy is measured in units of the mean level spacing d_0 in the spectrum of H_0, and time in units of the Heisenberg time $t_{\mathrm{H}} = 2\pi\hbar/d_0$. As a result, the variable t in (1) becomes dimensionless.

For our purpose it will prove convenient to write the perturbation in terms of a normalized eigenbasis:

$$W_\alpha = \sum_j w_j(\alpha)\, |v_j\rangle\langle v_j| \;, \tag{2}$$

where the orthonormal eigenstates $\{\,|v_j\rangle\,\}$ are assumed to be independent of α. This is normally well fulfilled in the case of global perturbations, and also in the case of many types of local perturbations, such as point like scatterer. A detailed discussion is given in section 3. In other words, (2) implies that $[W_\alpha, W_\beta] = 0$ for any α, β in the allowed range. Note that it is often possible to consider $H_0 + W_\alpha$ as the unperturbed Hamiltonian H_0'. Then, $W_\alpha' = 0$ and $W_\beta' = W_\beta - W_\alpha$ such that $[W_\alpha', W_\beta'] = 0$, trivially.

Returing to our original setup, we choose H_0 from one of the invariant ensembles, the Gaussian orthogonal ensemble (GOE) or the Gaussian unitary ensemble (GUE) [23]. Correspondingly, we assume that the perturbation W_α can be diagonalized either by an orthogonal (GOE case) or unitary (GUE case) transformation. In either case, we arrive at

$$H_\alpha = H_0 + \sum_j w_j(\alpha)\, |j\rangle\langle j| \;, \tag{3}$$

without changes in the random matrix ensemble for H_0. Here, the states $|i\rangle$ simply are the elements of the canonical basis of a complex vector space $\mathbb{C}^N$, where N may be assumed arbitrarily large but finite. In this situation, one may use the results of [13] to calculate the fidelity amplitude averaged over H_0 as the average of a certain scattering matrix correlation function within the framework of statistical scattering [10]. In what follows, we concentrate on the GOE case. The GUE case (which turns out to be even simpler) may be treated along similar lines, using [24].

2.2. Scattering matrix correlation functions

According to [10], the scattering matrix may be written as

$$S_{ab}(E) = \delta_{ab} - 2\mathrm{i}\pi\, V^\dagger\, \frac{1}{E - H_{\rm eff}}\, V \;, \tag{4}$$

where $H_{\rm eff} = H_0 - \mathrm{i}\pi\; V V^\dagger$ with H_0 from the GOE. The indices a, b denote the scattering channels which may be chosen in such a way that the column vectors of the rectangular matrix V are orthogonal. Analogous to (3) it is thus possible to diagonalize the perturbation such that

$$H_{\rm eff} = H_0 - \mathrm{i}\pi \sum_a \gamma_a\, |a\rangle\langle a| \tag{5}$$

where the parameters $\gamma_a > 0$ are the real and positive eigenvalues of $V V^\dagger$ (eigenstates corresponding to zero eigenvalues are ignored). According to [10], the average S-matrix (averaged over H_0), is then given as

$$\mathbb{E}(S_{ab}) = \frac{1 - \kappa_a}{1 + \kappa_a} \;, \qquad \kappa_a = \frac{\pi^2\, \gamma_a}{N} \;, \tag{6}$$

where we have assumed that the average level spacing for H_0 is equal to one. Here, we introduced the somewhat unusual notation $\mathbb{E}(\dots)$ for the ensemble average over the Gaussian random matrix ensembles, to avoid possible conflicts with the Bra-Ket notation used below. The main result of [10] consists in a triple integral which gives the spectral correlation function between different S-matrix elements

$$C[S_{ab}^*, S_{cd}](w) = \mathbb{E}[\, S_{ab}(E)^*\, S_{cd}(E+wd)\,] - \mathbb{E}[S_{ab}(E)\rangle^*]\, \mathbb{E}[S_{cd}(E+wd)]\ , \tag{7}$$

depending on the transmission coefficients $T_a = 4\kappa_a\,(1+\kappa_a)^{-2}$, only. Due to the convolution theorem, the Fourier transform of these correlation functions yields an average over different amplitudes of the evolution operator for the effective Hamiltonian $H_{\rm eff}$. Namely, for $t>0$:

$$\hat{C}[S_{ab}^*, S_{cd}](t) \propto \langle \hat{S}_{ab}(t)^*\, \hat{S}_{cd}(t)\rangle = \mathbb{E}(\, \langle b|\mathrm{e}^{2\pi\mathrm{i}H_{\rm eff}^\dagger\, t}|a\rangle\langle c|\mathrm{e}^{-2\pi\mathrm{i}H_{\rm eff}\, t}|d\rangle\,)\ , \tag{8}$$

It is still assumed that the Hamiltonian is written in the eigenbasis of the coupling $VV^\dagger$. Therefore, the states $|a\rangle, |b\rangle, |c\rangle$ and $|d\rangle$ represent elements of the canonical basis in $\mathbb{C}^N$. In what follows, we will only be concerned with the case when $c=a$ and $d=b$. This yields for the correlation function in (8):

$$\hat{C}[S_{ab}^*, S_{ab}](t) = \delta_{ab}\, T_a^2\, \langle Z\, J_a^2\rangle_I + (1+\delta_{ab})\, T_a T_b\, \langle Z\, P_{ab}\rangle_I\ , \tag{9}$$

where the angular brackets $\langle\dots\rangle_I$ denote the following weighted double-integral:

$$\langle\dots\rangle_I = \int_{\max(0,t-1)}^t \mathrm{d}r \int_0^r \mathrm{d}u\, \frac{(t-r)(r+1-t)}{(2u+1)(t^2-r^2+x^2)^2}\ , \tag{10}$$

and where we have introduced the following short hands:

$$Z = \prod_{j=1} \frac{1-T_j(t-r)}{\sqrt{1+2T_j\, r + T_j^2\, x^2}}\ , \quad x^2 = u^2\, \frac{2r+1}{2u+1}\ . \tag{11}$$

Note that changing the integration variable from u to x yields

$$2x\,\mathrm{d}x = (2r+1)\left[\frac{2u}{2u+1} - \frac{2u^2}{(2u+1)^2}\right]\mathrm{d}u \quad\Rightarrow\quad \frac{\mathrm{d}u}{2u+1} = \frac{\mathrm{d}x}{\sqrt{x^2+2r+1}}\ , \tag{12}$$

such that (10) may be written equivalently as

$$\langle\dots\rangle_I = \int_{\max(0,t-1)}^t \mathrm{d}r \int_0^r \mathrm{d}x\, \frac{(t-r)(r+1-t)}{\sqrt{x^2+2r+1}\,(t^2-r^2+x^2)^2}\ . \tag{13}$$

That expression can be compared directly to the results in [8, 9].

2.3. Connection to fidelity

Starting from (1), we insert the projection onto a random state $|b\rangle\langle b|$ into the definition of the fidelity amplitude:

$$f(t) \to f_{ab}(t) \propto \mathbb{E}(\, \langle a|\mathrm{e}^{2\pi\mathrm{i}H_\beta\, t}\, |b\rangle\langle b|\, \mathrm{e}^{-2\pi\mathrm{i}H_\alpha\, t}\, |a\rangle\,)\ , \tag{14}$$

where the average over the random state $|b\rangle\langle b|$ simply yields the identity times a normalization constant equal to the inverse Hilbert space dimension. As a result,

we obtain the product of two transition amplitudes which may be considered as a scattering fidelity as introduced in [11]. Comparing the effective Hamiltonian (5) with the perturbed Hamiltonian for a closed system as given in (3), we find that they share the same structure, and that we only need to allow the coupling parameters γ_a to become complex to unify both descriptions.

In [13], it has then been shown, that the analytical result for the correlation functions in (9) can be generalized to different effective Hamiltonians $H_{\rm eff}$ and $H'_{\rm eff}$, which differ only in the eigenvalues γ_a and γ'_a. In that case, one just needs to calculate the effective transmission coefficients

$$T_j = \frac{2\,(\kappa'_j + \kappa^*_j)}{(1+\kappa'_j)(1+\kappa^*_j)} \tag{15}$$

from the coupling parameters κ_a (corresponding to $H_{\rm eff}$) and κ'_a (corresponding to $H'_{\rm eff}$), as defined in (6). Then, the double integral in (10) yields the scattering fidelity, when replacing the transmission coefficients in the term Z by the effective transmission coefficients defined in (15).

Restricting ourselves to closed systems with Hermitian perturbations, we obtain from the comparison of (3) with (5) that $-\mathrm{i}\pi\gamma_j = w_j(\alpha)$, such that according to (6)

$$\kappa_j = \frac{\pi^2\,\gamma_j}{N} = \frac{\mathrm{i}\,\pi\, w_j(\alpha)}{N}\;, \qquad \kappa'_j = \frac{\mathrm{i}\,\pi\, w_j(\beta)}{N}\;. \tag{16}$$

Finally, to make sure that we really have a closed systemxs, we need the transmission coefficients T_a and T_b to be negligibly small. This means that the dynamics of the system is probed from the outside via scattering channels which are so weakly coupled to the system, that their effect on the dynamics is negligible. The functions to be integrated in (9) then become

$$\begin{aligned} J_a &\to 2t \\ P_{ab} &\to P_0 = 2\,[\,r^2 + (2r+1)\,t - t^2 - x^2\,]\;. \end{aligned} \tag{17}$$

Thereby, we obtain for the scattering fidelity

$$f_{ab}(\{\kappa_j\},\{\kappa'_j\}\,;\,t) \propto \delta_{ab}\,T_a^2\;4t^2\;\langle Z\rangle_I + (1+\delta_{ab})\;T_a\,T_b\;\langle Z\;P_0\rangle_I\;. \tag{18}$$

Here, we indicate explicitly the dependence of the scattering fidelity on the coupling parameters $\{\kappa_j\}$ and $\{\kappa'_j\}$ as their value will become important below, where we discuss normalization.

Normalization In order to calculate the fidelity amplitude from the scattering fidelity $f_{ab}(t)$, there is still the problem of normalization to be solved. This is because $f_{ab}(t)$ becomes an auto correlation function for $H_{\rm eff} = H'_{\rm eff}$, which still decays to zero in time, if the coupling to decay channels is finite. In [11], this problem has been solved by dividing the scattering fidelity through the geometric mean of the auto correlation functions of $H_{\rm eff}$ and $H'_{\rm eff}$. Below, we will see that this normalization procedure is somewhat simpler in the case of closed systems.

As mentioned earlier, when one is really interested in fidelity and $|b\rangle\langle b|$ has been inserted just for convenience as described in (14), one can normally assume that $a \neq b$. In addition, the case $a \neq b$ arises in the case of an explicit scattering fidelity experiment, where in- and out-going channels are chosen to be different (transmission measurement). Then, the formula for $f_{ab}(t)$ simplifies to

$$f_{ab}(t) \propto T_a T_b \, \langle P_0 \; Z\rangle_I \; . \tag{19}$$

In order to apply the normalization scheme from [11], we note that for the auto correlation functions:

$$f_{ab}(\{\kappa_j\}, \{\kappa_j\}\, ; \, t) = f_{ab}(\{\kappa'_j\}, \{\kappa'_j\}\, ; \, t) \propto T_a T_b \, \langle P_0\rangle_I \; . \tag{20}$$

This follows from the fact that $\kappa_j + \kappa_j^* = \kappa'_j + {\kappa'_j}^* = 0$ since the coupling parameters are purely imaginary in both cases. That implies that the effective transmission coefficients are zero, so that Z becomes equal to one. Since the auto correlation functions are the same, the geometric mean is also the same, and

$$f_{ab}(\{\kappa_j\}, \{\kappa'_j\}\, ; \, t) = \frac{T_a T_b \, \langle P_0 \; Z\rangle_I}{T_a T_b \, \langle P_0\rangle_I} = \frac{\langle P_0 \; Z\rangle_I}{\langle P_0\rangle_I} \; . \tag{21}$$

Now, it has been shown in [8] that for $Z = 1$, the resulting double integral yields

$$\langle P_0\rangle_I = \int_{\max(0,t-1)}^{t} \mathrm{d}r \int_0^r \mathrm{d}x \, \frac{(t-r)(r+1-t)\, P_0}{\sqrt{x^2+2r+1}\,(t^2-r^2+x^2)^2} = 1 \tag{22}$$

for any $t > 0$, so that

$$f_{ab}(\{\kappa_j\}, \{\kappa'_j\}\, ; \, t) = \langle P_0 \; Z\rangle_I \; . \tag{23}$$

This formula constitutes the first important result of our work, since it gives an exact analytical expression for the fidelity amplitude of a chaotic/diffusive wave system for an arbitrary perturbation.

In the special case, when the scattering fidelity is measured from a reflection amplitude ($a = b$), we find

$$f_{aa}(\{\kappa_j\}, \{\kappa'_j\}\, ; \, t) = N(t)^{-1} \, T_a^2 \, [\, 4t^2 \, \langle Z\rangle_I + 2 \, \langle Z \; P_0\rangle_I \,] \; , \tag{24}$$

where the geometric mean of the auto correlation functions, denoted by $N(t)$ turns out to be time dependent. While the effective transmission coefficients are still zero and $Z = 1$, the auto correlation functions now read:

$$f_{aa}(\{\kappa_j\}, \{\kappa_j\}\, ; \, t) = f_{aa}(\{\kappa'_j\}, \{\kappa'_j\}\, ; \, t) = N(t) = T_a^2 \, [\, 4t^2 \, \langle 1\rangle_I + 2 \, \langle P_0\rangle_I \,] \; . \tag{25}$$

The integral $\langle 1\rangle_I$ has been calculated in [25], with the result: $4t^2 \, \langle 1\rangle_I = 1 - b_2(t)$, where $b_2(t)$ is the two-point spectral form factor of the Gaussian orthogonal ensemble [23]. Hence, we obtain:

$$f_{aa}(\{\kappa_j\}, \{\kappa'_j\}\, ; \, t) = \frac{4t^2 \, \langle Z\rangle_I + 2 \, \langle Z \; P_0\rangle_I}{3 - b_2(t)} \; . \tag{26}$$

This result will be used in section 3.2, where we discuss experimental results for perturbations due to the displacement of an S-wave scatterer.

3. Local vs. global perturbations

A detailed discussion of the differences between local and global perturbations can be found in [22]. Consider (2), where a perturbation results in the change of several eigenvalues of the perturbation operator W. In order to affect the dynamics of the system (leading to the decay of the fidelity amplitude), one may either change only a few eigenvalues by a large amount (local perturbation) or very many eigenvalues by a small amount (global perturbation). Both cases are considered in this section.

3.1. Global perturbation

This was the first problem solved in the context of fidelity decay of quantum-chaotic systems [8, 9, 12]. Experimentally, the perturbation was realized in a chaotic microwave billiard by displacing one of the straight billiard boundaries. If described by (1) and (2), W_α may represent absolute displacements with respect to some initial position. Then, its eigenvector representation

$$W_\alpha = \sum_{j=1}^{N} w_j(\alpha)\, |v_j\rangle\langle v_j| \tag{27}$$

runs over a large number N of states. According to section 2.3, and in particular (15) and (16), the effective transmission coefficients become

$$\begin{aligned} T_j &= \frac{2\pi}{N}\, \frac{\mathrm{i}\, w_j(\beta) - \mathrm{i}\, w_j(\alpha)}{[1 + \mathrm{i}\pi\, w_j(\beta)/N]\, [1 - \mathrm{i}\pi\, w_j(\alpha)/N]} \\ &= 2\pi\mathrm{i}\, \delta_j\, (1 - \mathrm{i}\pi\, \delta_j) + \mathcal{O}\Big([w_j(\beta)/N]^3 ,\, [w_j(\alpha)/N]^3 \Big) , \end{aligned} \tag{28}$$

where $\delta_j = [w_j(\beta) - w_j(\alpha)]/N$. In this setting, global perturbations are characterized by the fact that the contribution of each individual term is negligible, while the perturbation becomes noticeable only because it is the sum of very many such contributions. This allows to perform a Taylor expansion of $\ln Z$ with respect to the coupling parameters δ_j. Starting from the Taylor expansion of Z_j with respect to the transmission coefficients

$$\begin{aligned} Z_j &= [1 - T_j\, (t - r)]\; [1 + 2T_j\, r + T_j^2\, x^2]^{-1/2} \\ &= [1 - T_j(t - r)]\; [1 - rT_j - (x^2 - 3r^2)\, T_j^2/2] + \mathcal{O}(T_j^3) \\ &= 1 - t\, T_j + [rt + r^2/2 - x^2/2]\; T_j^2 + \mathcal{O}(T_j^3) , \end{aligned} \tag{29}$$

we insert (28) into the above Taylor expansion, and obtain

$$\begin{aligned} \ln Z = -2\pi\mathrm{i} \sum_j \delta_j\, t - 2\pi^2 \sum_j (\, r^2 + (2r+1)\, t - t^2 - x^2\,)\, \delta_j^2 \\ + \mathcal{O}\Big([w_j(\beta)/N]^3 ,\, [w_j(\alpha)/N]^3 \Big) . \end{aligned} \tag{30}$$

In order to obtain a well defined function $Z(t, r, x)$ in the limit of $N \to \infty$, and vanishing perturbation: $\delta_j \to 0$, the parameters δ_j must scale with an appropriate negative power of N: (i) For $\delta_j \sim N^{-1}$, $\ln Z$ would converge to the finite value $-2\pi\mathrm{i} \sum_j \delta_j\, t$. (ii) For $\delta_j \sim N^{-1/2}$, the sum $\sum_j \delta_j$ could still converge, if the δ_j had different signs. In addition,

the sum $\sum_j {\delta_j}^2$ would always converge, while any higher order terms would vanish. (iii) For powers larger than $-1/2$, the sum $\sum_j {\delta_j}^2$ would always diverge, and the function $Z(t, r, x)$ would not be well defined. Hence, the cases (i) and (ii) are the only viable options, where

$$\sum_j \delta_j = \delta_{\rm s} \; , \qquad \sum_j {\delta_j}^2 = \lambda^2 \tag{31}$$

converge to finite values. This finally leads to

$$\lim_{N\to\infty} Z = \exp(-2\pi {\rm i}\, \delta_{\rm s}\, t - \pi^2 \lambda^2\, P_0) \; , \tag{32}$$

with P_0 given in (17). Note that by taking the absolute value squared of the fidelity amplitude, the dependence on $\delta_{\rm s}$ disappears and with it any possible problems with the convergence of this term.

To conclude this section about global perturbations, let us discuss the random matrix model for fidelity decay, as it has been first introduced in [7]. This model may be written as

$$H_\alpha = H_0 + \alpha\, V \; , \tag{33}$$

where the matrices H_0 and V are independent GOE matrices, normalized in such a way that the mean level spacing in the centre of the spectrum of H_0 is $d_0 = 1$, while for the perturbation matrix it holds

$$\langle V_{ij}\, V_{kl} \rangle = \delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl} \; . \tag{34}$$

Now, representing H_α in the eigenbasis of V, the perturbation becomes diagonal with eigenvalues $w_j(\alpha)$ showing a semi-circle distribution between $-2\alpha\sqrt{N}$ and $2\alpha\sqrt{N}$. Then, according to (2):

$$\kappa_j = 0 \; , \qquad \kappa'_j = \frac{{\rm i}\pi}{N}\, w_j(\alpha) \quad \Rightarrow \quad \delta_j = \frac{w_j(\alpha)}{N} \; , \tag{35}$$

which is of order $N^{-1/2}$, indeed. From the semi-circle distribution of the eigenvalues $w_j(\alpha)$ it follows that

$$\sum_j \delta_j = \frac{1}{N} \sum_j w_j(\alpha) = 0 \; , \qquad \sum_j {\delta_j}^2 = \frac{1}{N^2} \sum_j w_j(\alpha)^2 = \alpha^2 \; . \tag{36}$$

This shows that (32) applies for this case if we set $\delta_{\rm s} = 0$ and $\lambda = \alpha$. From (23) it then follows that

$$f_{ab}(t) = \Big\langle P_0\, {\rm e}^{-\pi^2\lambda^2\, P_0} \Big\rangle_I \; , \tag{37}$$

which agrees precisely with the result obtained in [8].

3.2. Local perturbations

In the case of local perturbations, W_α and W_β differ strongly in a relatively small subspace. A Taylor series expansion as in the previous case is therefore not useful, and it is also less likely that $[W_\alpha, W_\beta] = 0$. Thus, it seems necessary to redefine the perturbation by considering H_α as the unperturbed system and $W_\beta - W_\alpha$ as the perturbation. In doing so, it is assumed that including W_α into H_0 doesn't change its statistical (i.e. random matrix) properties. We may then choose a basis in which $W_\beta - W_\alpha$ is diagonal, the transformation into that basis leaves the random matrix ensemble for the new H_0 invariant, so that we arrive again at a description in which the perturbation is diagonal.

In contrast to the previous case, we have now only a small number of non-zero diagonal elements, while each of them may be very large. In principle, each element alone can cause the fidelity to decay as fast as in the case of a global perturbation. In what follows, we restrict ourselves to the least biased case, where the states $|a\rangle$ and $|b\rangle$ coupled to the measurement channels [cf. (8) and (9)] are not involved into the perturbation. In this case, the results become again independent from the measurement channels, and one can then repeat the previous argument to show that scattering fidelity (if measured in transmission) and fidelity amplitude must coincide.

3.2.1. Moving scatterer This case refers to the displacement of a small scatterer from a position $\vec{r}_1$ to another position $\vec{r}_2$. As explained above, this is modelled by the unperturbed Hamiltonian consisting of the system with scatterer at position $\vec{r}_1$, while the perturbation consists in removing the scatterer from position $\vec{r}_1$ and placing it at position $\vec{r}_2$. For a point-like scatterer, the effect of the scatterer can be described by one single state (the perturbation operator corresponding to that scatterer has only one non-zero eigenvalue). Therefore,

$$H_\alpha = H_0 \; , \qquad H_\beta = H_0 + w(\beta)\, (\, |v_2\rangle\langle v_2| - |v_1\rangle\langle v_1|\,) \; . \tag{38}$$

For the scattering approach to fidelity, this means that the perturbation must be described by two effective transmission coefficients:

$$T_1 = \frac{-2\pi \mathrm{i}\, \delta_1}{1 - \mathrm{i}\pi\, \delta_1} \; , \quad T_2 = \frac{2\pi \mathrm{i}\, \delta_1}{1 + \mathrm{i}\pi\, \delta_1} \; , \qquad \delta_1 = \frac{w(\beta)}{N} \; . \tag{39}$$

Here, δ_1 may become arbitrarily large so that T_1 as a function of δ_1 traces a path in the complex plane which starts at $T_1 = 0$ and ends at $T_1 = 2$, while $T_2 = T_1^*$. Then

$$Z = \prod_{j=1}^{M} \frac{1 - T_j(t-r)}{\sqrt{1 + 2T_j r + T_j^2 x^2}} = \frac{|1 - T_1\,(t-r)|^2}{|1 + 2T_1\, r + T_1^2 x^2|} \tag{40}$$

inserted into (23) or (26) yields the exact analytical expression for the decay of the fidelity amplitude.

3.2.2. Comparison with experiment The perturbation we just described, applies precisely to an experiment published in [20, 21]. There, a small disk of diameter 4.6 mm has been moved in steps of $|\Delta r| = 1$ mm through a rectangular two-dimensional microwave billiard with 19 additional random scatterers. Then, the reflection spectrum has been measured for 300 different positions of the moving disk in a frequency range from 3.5 to 6 GHz. In this frequency range, it was still possible to extract resonance positions and amplitudes by Lorentzian fits. The statistical properties of the spectrum as well as the wave functions were in agreement with the random matrix expectation for quantum chaotic or weakly disordered systems.

From Berry's model of the random superposition of plane waves [26], it is possible to obtain a connection between the displacement $|\Delta r|$ of the movable disk and the parameter δ_1 which measures the perturbation strength. Translating the corresponding equation from [21] to our system of units and parameters, we obtain

$$\delta_1 = \frac{\alpha}{4}\,\sqrt{1 - J_0(k|\Delta r|)^2}\;, \tag{41}$$

where α is a dimensionless factor related to the electromagnetic properties of the movable disk, which can be determined independently (*e.g.* from the variance of the level velocities). For the wavenumber k, we choose a value which corrresonds to the frequency in the centre of the range considered, which yields

$$k = \frac{2\pi\, f}{c} = 0.996\,\mathrm{cm}^{-1}\;. \tag{42}$$

For the displacements considered in [21], the relation between δ_1 and $|\Delta r|$ is still approximately linear, as can be seen from the fact that $k\,|\Delta r|$ is small as compared to one in all cases (see the captions of figure 1). Finally, we find that $\alpha = 1$ provides the best agreement between the theory and experiment. Using time independent perturbation theory, the authors of [21] obtained for the decay of the fidelity amplitude the following asymptotic result:

$$f(t) = \frac{1}{\sqrt{1 + (4\delta_1 t)^2}}\;, \tag{43}$$

valid for finite $\delta_1\, t$, in the case where $\delta_1 \to 0$ and $t \to \infty$, and in agreement with our asymptotic result (61), derived below (section 4).

In figure 1 we show the experimental data for the decay of the absolute value squared $|f_{aa}(t)|^2$ as obtained in [21]. In the experiment, this quantity is obtained from ensemble averages of the respective correlation functions, introduced in (14). The results are compared to the theoretical predictions based on the perturbative result, (43), and on our exact analytical expression for a reflection measurement, (26).

We focus here on the behaviour of the fidelity at small times, where the asymptotic formula is expected to be less accurate, and indeed, we find significant deviations for the cases $\delta_1 \approx 0.14$ (green points vs. dashed green line) and $\delta_1 \approx 0.28$ (blue points vs. dashed blue line). For these cases, the experimental fidelity decay has a notable linear component at small times, which cannot be reproduced by the perturbative formula, (43). By contrast, our exact analytical result contains that linear component

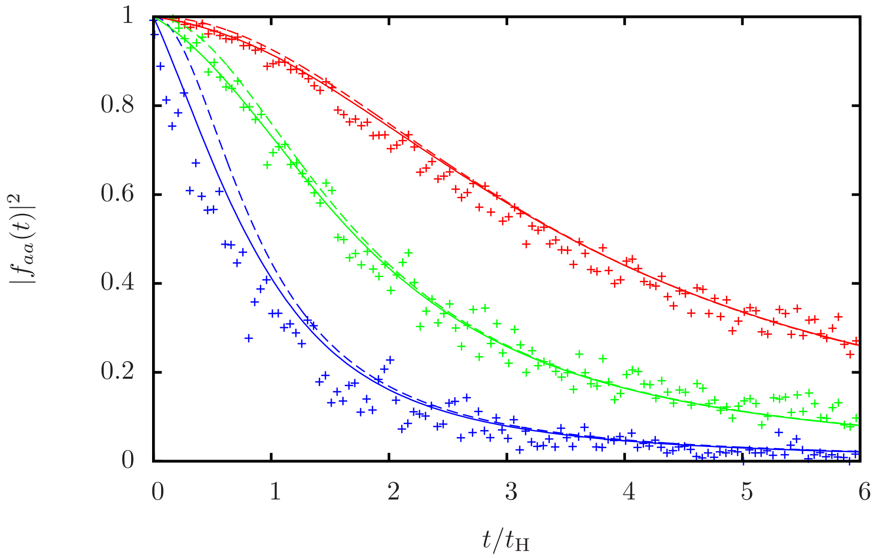

Figure 1. Experimental data for the fidelity decay due to a moving scatterer from [21], compared to the approximate (dashed lines) and to the exact theory (solid lines). The different colours, red, green, and blue, correspond to different displacements $\Delta r = 1\,\mathrm{mm}$ ($\delta_1 \approx 0.07$), $2\,\mathrm{mm}$ ($\delta_1 \approx 0.14$), and $4\,\mathrm{mm}$ ($\delta_1 \approx 0.28$), respectively.

and therefore agrees much better with the experiment (solid lines). Still, some differences remain for $\delta_1 \approx 0.28$. We believe that these are due to problems on the experimental side. One error source consists in the wide frequency range used, which, according to (41) leads to a considerable variation in the perturbation strength. Another problem is related to the upper end of the frequency range, which implies rather small wavelengths, for which the scatterer to be moved may no longer be point like. For a more significant test of our analytical formula, one would need a different experimental design, providing higher accuracies at strong perturbations in the vicinity of the Heisenberg time.

In the perturbative result, (43), it makes no difference whether the measurement is performed as a transmission measurement with two measurement antennas or as a reflection measurement with only one. As we have seen from (23) and (26), for the exact analytical result this is no longer true. In figure 2, we compare both cases for four different perturbation strengths. The results show that the difference $|f_{aa}(t)|^2 - |f_{ab}(t)|^2$ is clearly present, but usually quite small. For very large ($\delta_1 = 0.56$, narrow peak at $t \lesssim t_H$) as well as for very small perturbation strengths, it seems that the difference tends to disappear. This is consistent with our treatment of the perturbative case in Sec. 4.4.

4. Perturbative regime

In the perturbative regime, the fidelity amplitude of (1) can be calculated using first order time-independent perturbation theory [2]. For $\{|j\rangle\}$ denoting the eigenbasis of H_β

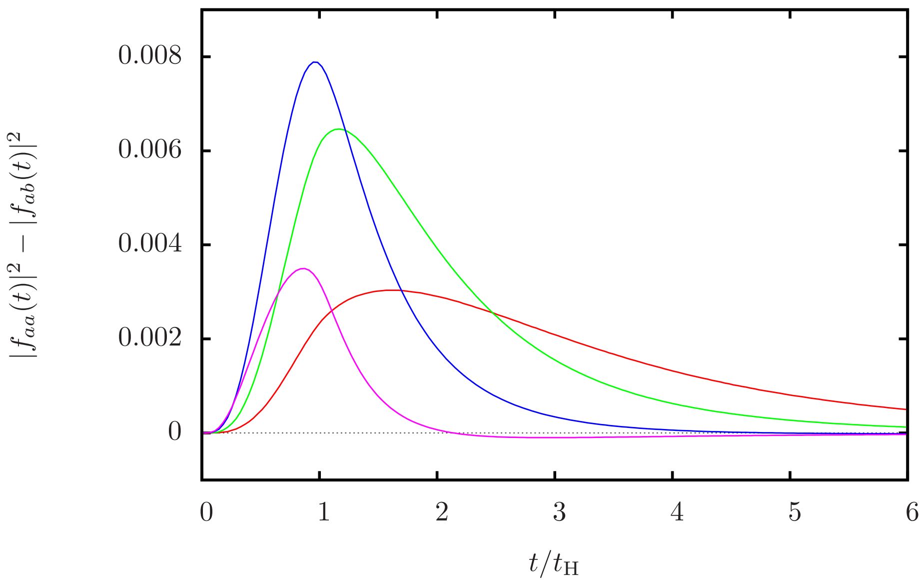

Figure 2. Comparison between fidelity decay, measured in reflection $|f_{aa}(t)|^2$ and in transmission $|f_{ab}(t)|^2$. Different colours correspond to different perturbation strengths, $\delta_1 = 0.07$ (red line), 0.14 (green line), 0.28 (blue line), and 0.56 (pink line).

and $V = H_\alpha - H_\beta$, we then find

$$f(t) \approx \sum_j \langle\alpha|j\rangle\, {\rm e}^{2\pi {\rm i} E_j(\beta)\, t}\, {\rm e}^{-2\pi{\rm i}[E_j(\beta) + \langle j|V|j\rangle]\, t}\, \langle j|\alpha\rangle = \sum_j |\langle j|\alpha\rangle|^2\, {\rm e}^{-2\pi{\rm i}\langle j|V|j\rangle\, t}\ , \tag{44}$$

where $|\alpha\rangle$ denotes some initial state, and $\{E_j(\beta)\}$ denote the eigenvalues of H_β. This expression shows that in the perturbative regime the fidelity decay depends on the product between time and perturbation strength. The perturbative result becomes exact only in the limit of vanishing perturbation strength. To yield a finite value for the fidelity amplitude, time must then tend to infinity such that the product between perturbation strength and time remains constant. We therefore define the perturbative regime as the limit

$$t \to \infty\ , \quad \forall j\ :\ T_j \to 0\ , \tag{45}$$

such that $t \sum T_j$ and $t^2 \sum_j T_j^2$ remain both finite. In what follows, we calculate $f_{ab}(t)$ and $f_{aa}(t)$ in that limit, starting from the exact analytical expressions (23) and (26), via an asymptotic expansion of the respective integrals. This is done in two steps.

4.1. Step one

Here, we will demonstrate that

$$\langle\ldots\rangle_I \sim \langle\ldots\rangle_A\ , \quad \text{with} \quad \langle\ldots\rangle_A = \int_{t-1}^{t} {\rm d}r \int_0^{\sqrt{r}} {\rm d}u\, \frac{(t-r)(r+1-t)}{(2u+1)(t^2-r^2+x^2)^2}\ , \tag{46}$$

where the ellipsis above may be replaced by either $Z\,P_0$ or $4t^2\,Z$. Here and in what follows, the symbol $\sim$ denotes the perturbative limit we are interested in.

For the first case, our claim follows from

$$\int_{t-1}^{t} \mathrm{d}r \int_{\sqrt{r}}^{r} \mathrm{d}u \, \frac{(t-r)(r+1-t)\; P_0\, Z}{(2u+1)(t^2-r^2+x^2)^2} \sim 0\,. \tag{47}$$

Since $P_0 > 0$ and $0 < Z < 1$ in the whole region of integration, it is sufficient to show that (47) holds for $Z = 1$. Furthermore, since we can maximize $(t-r)(r+1-t)$ in the interval $t-1 < r < t$ by $1/4$, it is sufficient to prove that

$$\max_{t-1<r<t} \int_{\sqrt{r}}^{r} \mathrm{d}u \, \frac{r^2+(2r+1)t-t^2-x^2}{(2u+1)(t^2-r^2+x^2)^2} \sim 0\,, \tag{48}$$

where we have used that $P_0 = 2\,(r^2+(2r+1)t-t^2-x^2)$. Denoting this integral with J, we realize that

$$J < \int_{\sqrt{r}}^{r} \mathrm{d}u \, \frac{r^2+(2r+1)t-t^2}{(2u+1)(t^2-r^2+x^2)^2} \tag{49}$$

Then, because $r^2+(2r+1)t-t^2 = t+2r^2-(r^2-2rt+t^2)$,

$$J < \int_{\sqrt{r}}^{r} \mathrm{d}u \, \frac{2t^2+t}{(2u+1)x^4} = (2t^2+t) \int_{\sqrt{r}}^{r} \mathrm{d}u \, \frac{2u+1}{(2r+1)^2\, u^4} \tag{50}$$

Evaluating the last integral we finally obtain:

$$J < \frac{t\,(2t+1)}{(2r+1)^2} \left(\frac{3\sqrt{r}+1}{3r^{3/2}} - \frac{3r+1}{3r^3} \right) = 0\,, \tag{51}$$

which completes the proof. For the second case, we replace P_0 with $2t^2$ one arrives at the same result, which may be seen from (50).

4.2. Step two

According to (46) the perturbative limit only requires integration of u up to $u = \sqrt{r}$. This implies that

$$Z = \prod_j \frac{1-T_j(t-r)}{\sqrt{1+2T_j\, r+T_j^2\, x^2}} \sim \prod_j \frac{1}{\sqrt{1+2T_j\, t}}\,, \tag{52}$$

since $(t-r)$ is of order one, $t-1<r<t$, and

$$T_j^2\, x^2 = T_j^2\, u^2\, \frac{2r+1}{2u+1} < T_j^2\, r\, \frac{2r+1}{2\sqrt{r}+1} \sim 0\,. \tag{53}$$

Therefore, we obtain for $f_{ab}(t) = \langle Z\, P_0 \rangle_I$:

$$f_{ab}(t) \sim \prod_j \frac{1}{\sqrt{1+2T_j\, t}} \langle P_0 \rangle_A \sim \prod_j \frac{1}{\sqrt{1+2T_j\, t}}\,. \tag{54}$$

This simply follows from the fact that $1 = \langle P_0 \rangle_I \sim \langle P_0 \rangle_A$. For the scattering fidelity in a reflection measurement, we obtain

$$f_{aa}(t) \sim \frac{4t^2\, \langle Z \rangle_I + 2\, \langle Z\, P_0 \rangle_I}{3}\,, \tag{55}$$

since $b_2(t)$ in (26) tends to zero for large times. Here, it only remains to treat the first term in the nominator:

$$4t^2\,\langle Z\rangle_I \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}\; 4t^2\,\langle 1\rangle_A \sim \prod_j \frac{1}{\sqrt{1+2T_j\,t}}\;, \tag{56}$$

since $1 = 4t^2\,\langle 1\rangle_I \sim 4t^2\,\langle 1\rangle_A$. Thus, in the perturbative regime, we obtain the same result no matter whether we perform a transmission or a reflection measurement:

$$f_{aa}(t) \sim f_{ab}(t) \sim f_{\rm pert}(t) = \prod_j \frac{1}{\sqrt{1+2T_j\,t}}\;. \tag{57}$$

4.3. Global perturbation

Global perturbations are discussed in detail in Sec. 3.1, where (28) relates the effective transmission coefficients with the perturbation strengths δ_j. Taking also (31) into account, we may write for $f(t)$ up to second order in the perturbation strength:

$$\ln\, f_{\rm pert}(t) \sim -\frac{1}{2}\sum_j \ln\left[\,1+4\pi{\rm i}\,\delta_j\,(1-{\rm i}\pi\,\delta_j)\,t\,\right] \sim -\sum_j \left[\,2\pi{\rm i}\,\delta_j\,t + 2\pi^2\delta_j^2\,(2t^2+t)\,\right]. \tag{58}$$

Since we are working in the perturbative regime, where t goes as fast to infinity as the δ_j go to zero, terms containing $\delta_j^2\,t$ can be neglected. Finally, we obtain

$$\ln\, f_{\rm pert}(t) \sim {\rm e}^{-2\pi{\rm i}\,\delta_{\rm s}\,t - 4\pi^2\lambda^2\,t^2}\;. \tag{59}$$

4.4. Moving scatterer

Inserting the effective transmission coefficients from (39) describing a moving scatterer into (57), we find

$$f_{\rm pert}(t) = \left| 1 - \frac{4{\rm i}\,\delta_1\,t}{1-{\rm i}\,\delta_1}\right|^{-1}\;. \tag{60}$$

In the perturbative limit considered here, means that $T_1 \to 0$, $t\to\infty$ such that $T_1\,t$ remains constant. This implies however that also $\delta_1 \to 0$ such that $\delta_1\,t$ remains constant. Therefore

$$f_{\rm pert}(t) \sim |1-4{\rm i}\,\delta_1\,t|^{-1} = \frac{1}{\sqrt{1+(4\delta_1\,t)^2}}\;. \tag{61}$$

5. Comparison with parametric level correlations

For any quantum mechanical model of the form

$$H(\lambda) = H_0 + \lambda\, V\;, \tag{62}$$

we may consider the level dynamics obtained from plotting the eigenvalues of $H(\lambda)$ as functions of λ. For convenience, we assume here again that for any value of λ, the average level spacing is one. In a typical random matrix model, one would eventually choose V from a Gaussian random ensemble with non-diagonal elements of unit variance.

The parametric level correlations $X(\lambda, r)$ describe the probability to find two eigenvalues, one of $H(0)$ and the other one of $H(\lambda)$ at a distance r. The quantity to be compared to the fidelity amplitude is the Fourier transform of the parametric level correlations [14]:

$$K(\lambda, t) = \int \mathrm{d}r\; \mathrm{e}^{2\pi \mathrm{i}\, rt}\, [\, X(\lambda, r) - 1\,]\,. \tag{63}$$

Note that for $\lambda \to 0$, this quantity converges to the complement of the two-point form factor: $K(0, t) = 1 - b_2(t)$ [27, 23].‡ The relation discussed in [14] is a relation between the parametric level correlations on the one hand, and the fidelity amplitude $f_\lambda(t) = f(\lambda, t)$ on the other. It may be expressed as

$$f(\lambda, t) = \frac{-\beta}{4\pi^2 t^2}\, \frac{\partial}{\partial\, (\lambda^2)}\, K(\lambda, t)\,, \tag{64}$$

with β being the Dyson parameter [28] which is one in our case. We consider systems with an anti-unitary symmetry such as time reversal invariance.

5.1. Global perturbation

From [22] we find

$$X(\lambda, r) = 1 + \mathrm{Re} \iint_1^\infty \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \int_{-1}^1 \mathrm{d}\mu'\; \frac{(\lambda_1\lambda_2 - \mu')^2\, (1 - \mu'^2)\, \mathrm{e}^{\mathrm{i}\pi\, r_+\, (\lambda_1\lambda_2 - \mu')}\, \mathrm{e}^{-\pi^2\lambda^2\, P_0}}{(1 + 2\lambda_1\lambda_2\, \mu' - \lambda_1^2 - \lambda_2^2 - \mu'^2)^2}\,, \tag{65}$$

where $2P_0 = 1 + 2\lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \mu'^2$. We will see below, that this quantity is precisely the same as P_0 defined in the previous section, in (17). The first substitution, $\mu' \to \mu = (\lambda_1\lambda_2 - \mu')/2$, yields

$$X(\lambda, r) = 1 + 2\ \mathrm{Re} \iint_1^\infty \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \int_{(\lambda_1\lambda_2 - 1)/2}^{(\lambda_1\lambda_2 + 1)/2} \mathrm{d}\mu\; \frac{4\mu^2\, (1 - \mu'^2)\, \mathrm{e}^{2\mathrm{i}\pi\, r_+\, \mu}\, \mathrm{e}^{-\pi^2\lambda^2\, P_0}}{(1 + 2\lambda_1\lambda_2\, \mu' - \lambda_1^2 - \lambda_2^2 - \mu'^2)^2}\,. \tag{66}$$

In order to shorten the expressions, we keep writing μ' which must be understood as being a function of μ. Now, we can switch to the Fourier transform, which turns the Fourier factors in delta functions:

$$K(\lambda, t) = \iint_1^\infty \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \int_{(\lambda_1\lambda_2 - 1)/2}^{(\lambda_1\lambda_2 + 1)/2} \mathrm{d}\mu \frac{[\, \delta(t + \mu) + \delta(t - \mu)\,]\; 4\mu^2\, (1 - \mu'^2)\, \mathrm{e}^{-\pi^2\lambda^2\, P_0}}{(1 + 2\lambda_1\lambda_2\, \mu' - \lambda_1^2 - \lambda_2^2 - \mu'^2)^2}\,. \tag{67}$$

This shows that the function $K(\lambda, t)$ is symmetric in time. In what follows, we thus assume $t > 0$. The remaining delta function already allows to eliminate the μ-integration. However, before actually doing so, we perform a variable transformation on the λ_1, λ_2 integrals:

$$(\lambda_1, \lambda_2) \to (r', x')\,, \qquad r' = \lambda_1\lambda_2\,, \quad x' = \lambda_2 - \lambda_1 \tag{68}$$

‡ The definition of $K(\lambda, t)$ in [14] uses the wrong sign, while the final expressions for $K_1(\lambda, t)$ misses the variable v in the nominator of the integrand.

The Jacobian of this transformation is simply $J = (\lambda_1 + \lambda_2)^{-1} = (x'^2 + 4r')^{-1/2}$. Therefore,

$$K(\lambda, t) = \int_1^\infty \mathrm{d}r' \int_{1-r'}^{r'-1} \frac{\mathrm{d}x'}{\sqrt{x'^2 + 4r'}} \int_{(r'-1)/2}^{(r'+1)/2} \mathrm{d}\mu \\ \frac{4\mu^2\,(1 - r' + 2\mu)(1 + r' - 2\mu)\,\delta(t-\mu)\,\mathrm{e}^{-\pi^2\lambda^2 P_0}}{[1 + 2r'(r' - 2\mu) - x'^2 - 2r' - (r' - 2\mu)^2]^2} \tag{69}$$

For the μ-integral not to yield zero, it must hold that $(r'-1)/2 < t < (r'+1)/2$. This modifies the limits of the r'-integral as follows:

$$K(\lambda, t) = 4t^2 \int_{\max(1,2t-1)}^{2t+1} \mathrm{d}r' \int_{1-r'}^{r'-1} \frac{\mathrm{d}x'}{\sqrt{x'^2 + 4r'}}\, \frac{(1 - r' + 2t)(1 + r' - 2t)\,\mathrm{e}^{-\pi^2\lambda^2 P_0}}{[1 + 2r'(r' - 2t) - x'^2 - 2r' - (r' - 2t)^2]^2} \tag{70}$$

Further substitutions: $r' = 2r + 1$ and $x' = 2x$ and the fact that the integrand is a symmetric function of x, yield

$$K(\lambda, t) = 4\,t^2 \int_{\max(0,t-1)}^{t} \mathrm{d}r \int_0^r \frac{\mathrm{d}x\,(t-r)\,(r+1-t)\,\mathrm{e}^{-\pi^2\lambda^2 P_0}}{\sqrt{x^2 + 2r + 1}\,(t^2 - r^2 + x^2)^2} = 4\,t^2\, \left\langle \mathrm{e}^{-\pi^2\lambda^2 P_0} \right\rangle_I \,. \tag{71}$$

Comparing to (37), it is now easily checked that $K(\lambda, t)$ as defined here fulfils the fidelity amplitude – parametric form factor relation (64).

5.2. General perturbation

In [22] it is shown that parametric level correlations can be calculated for arbitrary perturbations. According to this reference, the term describing the global perturbation $\sigma_{\rm glob} = \pi^2\lambda^2 P_0$ must be replaced by $\sigma = \sigma_{\rm glob} + \sigma_{\rm loc}$, where

$$\sigma_{\rm loc}(\lambda_1, \lambda_2, \mu') = \frac{1}{2} \sum_j \ln\left[\frac{1 + 2\mathrm{i}\,\kappa'_j\,\lambda_1\lambda_2 - \kappa'^{\,2}_j\,(\lambda_1^2 + \lambda_2^2 - 1)}{(1 + \mathrm{i}\,\kappa'_j\,\mu')^2} \right] \,. \tag{72}$$

Note that the discussion in Sec. 3.1 shows that the additional global perturbation could always be incorporated into $\sigma_{\rm loc}$, via a large number of additional channels with infinitesimal perturbations. However, in order to establish the desired relation between fidelity decay and the parametric level correlations, it is important to have the parameter λ describing the global perturbation at hand.

Now, we should go through the calculation of $K(\lambda, t)$ again, replacing $\sigma_{\rm glob}$ in (66) with the more general expression σ. As a consequence, the integrand is no longer real, which affects (67). While the delta function $\delta(t-\mu)$ is multiplied with the same term as before, the second delta function $\delta(t+\mu)$ is now multiplied with its complex conjugate. Therefore $K(\lambda, t)$ is no longer symmetric. Instead $K(\lambda, t) = K(\lambda, -t)^*$, which nevertheless allows to continue the calculation without changes for $t > 0$. Only at (71) we need to express $\sigma_{\rm loc}$ in the current integration variables. That results in

$$\exp\left[\, -\sigma_{\rm loc}(r, x, t)\,\right] = \prod_j \frac{1 - T_j\,(t - r)}{\sqrt{1 + 2T_j\,r + T_j^2\,x^2}} \,, \tag{73}$$

where $T_j = 2\mathrm{i}\kappa'_j/(1+\mathrm{i}\,\kappa'_j)$, just as in (28). Inserting this expression into (71) and comparing to the general result (23) for the decay of the fidelity amplitude ($a \neq b$) we find that the following relation holds:

$$f_{ab}(\{\kappa'_j\},t) = \frac{-1}{4\pi^2\,t^2}\;\frac{\partial}{\partial(\lambda^2)}\,K(\{\kappa'_j\},\lambda,t)\bigg|_{\lambda=0} \;. \tag{74}$$

The fact that any global perturbation may always be modelled with a large number of additional terms in $\sigma_{\rm loc}$, allows for a final slight generalization:

$$f_{ab}(\{\kappa'_j\},\lambda_0,t) = \frac{-1}{4\pi^2\,t^2}\;\frac{\partial}{\partial(\lambda^2)}\,K(\{\kappa'_j\},\lambda,t)\bigg|_{\lambda=\lambda_0} \;. \tag{75}$$

This relation constitutes our second important result. In practice, this relation means that one can obtain the fidelity amplitude, by measuring the change of the parametric level correlations under the increment of a global perturbation.

6. Conclusions

In this paper, we have used the results of [13] to derive an exact analytical expression for the fidelity decay in a closed chaotic/diffusive wave system, under arbitrary Hermitian perturbations. For illustration, we used that result to re-derive the known formula for the fidelity decay in the case of a global perturbation [8, 9]. In a second application, we calculated the fidelity amplitude for a moving S-wave scatterer, and checked that it describes corresponding experimental results reported in [21] well. Finally, we generalized a relation between the fidelity amplitude and parametric level correlations introduced in [14] to arbitrary perturbations.

In the present work, we restricted ourselves to matrix ensembles based on the Gaussian orthogonal ensemble (GOE). For the comparison with experimental data, this is the most important case. However, our results can also be translated to the Gaussian unitary ensemble (GUE), using the analog of the VWZ-formula published in [24]. For the Gaussian symplectic ensemble (GSE), the corresponding analytical expressions for the parametric level correlations and the correlations between scattering matrix elements are unfortunately not yet available, but we would still expect a similar relation to hold.

It would be interesting to perform an experiment similar to the one analyzed in [20, 21], in order to verify our results with higher accuracy and for larger perturbation strengths. Particularly interesting would be the regime, where the perturbation strength depends in a non-linear way on the displacement of the scatterer, see (41). If the microwave experiment would allow to measure fidelity decay and parametric level correlations at the same time, one could test the applicability of the relation (75) between both quantities practice. Finally, one may intend to generalize (75) further to scattering systems and non-Hermitian perturbations (*e.g.* coupling fidelity).

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